Institute of Media,
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Wavelet Bases

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Wavelet Bases

- Orthogonal Wavelet Bases
- Classes of Wavelet Bases
- Wavelets and Filter Banks
- Biorthogonal Wavelet Bases

Wavelet bases

Introduction

- While browsing webpages, you certainly have downloaded interlaced GIF images. During the download, a progressively detailed image is displayed on screen. This idea of consecutive approximations at finer and finer resolutions is formalized by the concept of multiresolution approximation (or multiresolution analysis).

- Dyadic wavelets are wavelets which satisfy an additional scaling property. This property allows the implementation of a Fast Dyadic Wavelet Transform with filter banks.

- The definition of dyadic wavelets comes from the definition of multiresolution approximations.

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Wavelet bases

Multiresolution Approximations

A sequence of closed subspaces $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of $\mathbf{L}^2(\mathbb{R})$:

obtained by dilation $f(t) \in \mathbf{V}_j \Leftrightarrow f\left(\frac{t}{2}\right) \in \mathbf{V}_{j+1}$

stable under dyadic translation $f(t) \in \mathbf{V}_j \Leftrightarrow f\left(t - 2^j k\right) \in \mathbf{V}_j$

$$\boxed{\mathbf{L}^2(\mathbb{R})} \longrightarrow \cdots \longrightarrow \mathbf{V}_{j-1} \longrightarrow \mathbf{V}_j \longrightarrow \mathbf{V}_{j+1} \longrightarrow \cdots \longrightarrow \boxed{\{0\}}$$

$$\boxed{\lim_{j\to-\infty} \left\| f - P_{\mathbf{V}_j} f \right\| = 0} \longrightarrow \boxed{\lim_{j\to+\infty} \left\| P_{\mathbf{V}_j} f \right\| = 0}$$

Definition 7.1: Multiresolutions A sequence $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $\mathbf{L}^2(\mathbb{R})$ is a multiresolution approximation if the above properties are satisfied and there exists θ such that $\{\theta(t-n)\}_{n\in\mathbb{Z}}$ is a Riesz basis of $\mathbf{V}_0$.

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Scaling Function

Theorem 7.1. Let $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$ be a multiresolution approximation and ϕ be the scaling function having a Fourier transform

$$\hat{\phi}(\omega)=\frac{\hat{\theta}(\omega)}{\left(\sum_{k=-\infty}^{+\infty}\left|\hat{\theta}(\omega+2k\pi)\right|^2\right)^{1/2}}$$

Let us denote

$$\phi_{j,n}(t)=\frac{1}{\sqrt{2^j}}\phi\left(\frac{t-2^j n}{2^j}\right)$$

The family $\{\phi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{V}_j$ for all $j\in\mathbb{Z}$

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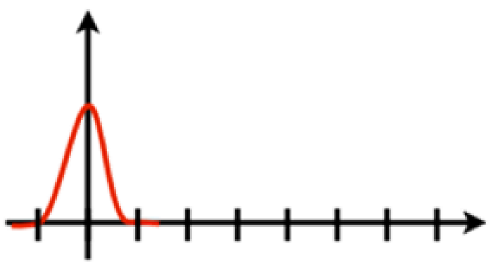
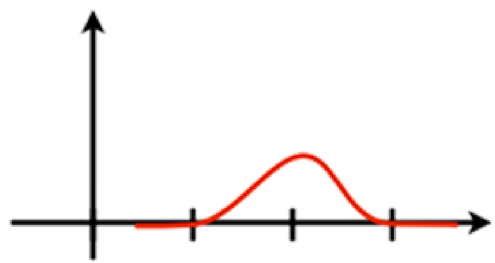
Scaling Function

Scaling function ϕ: $\quad \mathbf{V}_j = \mathrm{Span}\{\phi_{j,n}\}_{n\in\mathbb{Z}} \quad \phi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\phi\left(\frac{t-2^j n}{2^j}\right)$

$\phi = \phi_{0,0} \in \mathbf{V}_0$



$\phi_{1,n} \in \mathbf{V}_1$

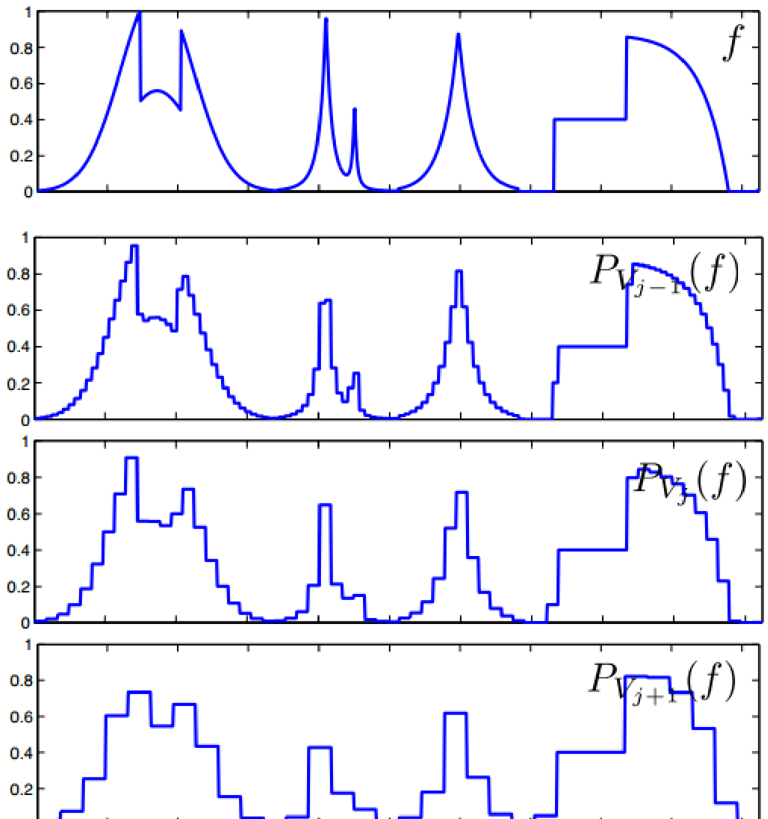


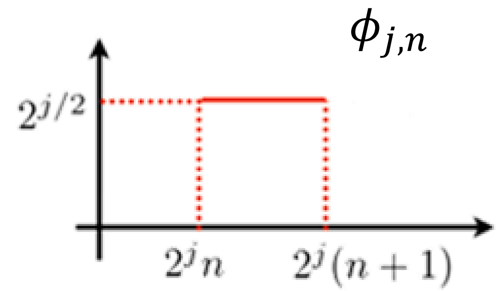
$\phi_{-1,n} \in \mathbf{V}_{-1}$

Example: Haar Multiresolutions

Piecewise-constant approximation: $\mathbf{V}_j = \{f \mid f \text{ constant on } [2^j n, 2^j(n+1)]\}$

Linear approximation of f using $\mathbf{V}_j$: $P_{\mathbf{V}_j}(f) = \sum_n \langle f, \phi_{j,n} \rangle \phi_{j,n}$



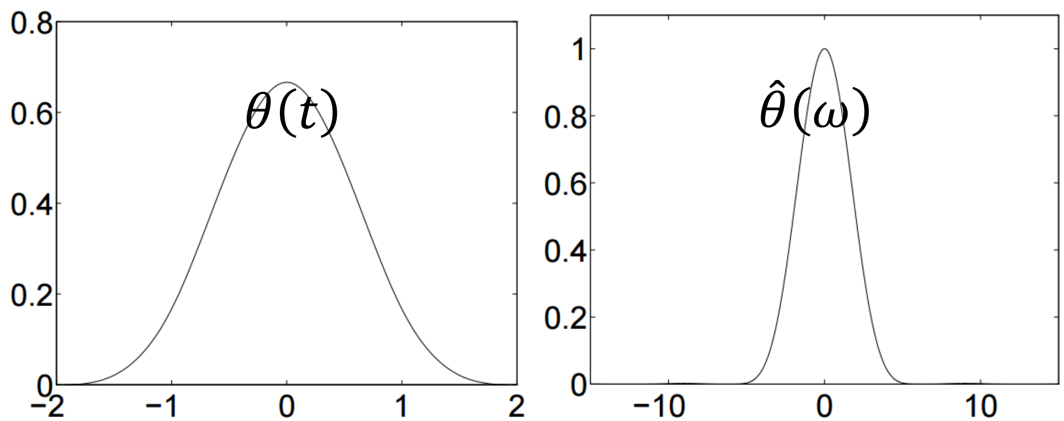


Generalizes to higher order spline approximations.

Example: Spline Approximations

- The space $\mathbf{V}_j$ of splines of degree $m \geq 0$ is the set of functions that are $m-1$ times continuously differentiable and equal to a polynomial of degree m on any interval $\left[2^j n, 2^j(n+1)\right]$ for $n \in \mathbb{Z}$. When $m=0$, it is a piecewise constant multiresolution approximation. When $m=1$, functions in $\mathbf{V}_j$ are piecewise continuous.

- A Riesz basis of polynomial splines is constructed with *box splines*. A box spline θ of degree m is computed by convolving the box window $\mathbf{1}_{[0,1]}$ with itself $m+1$ times and centering at 0 or ½.



Cubic box Spline θ and its Fourier transform $\hat{\theta}$

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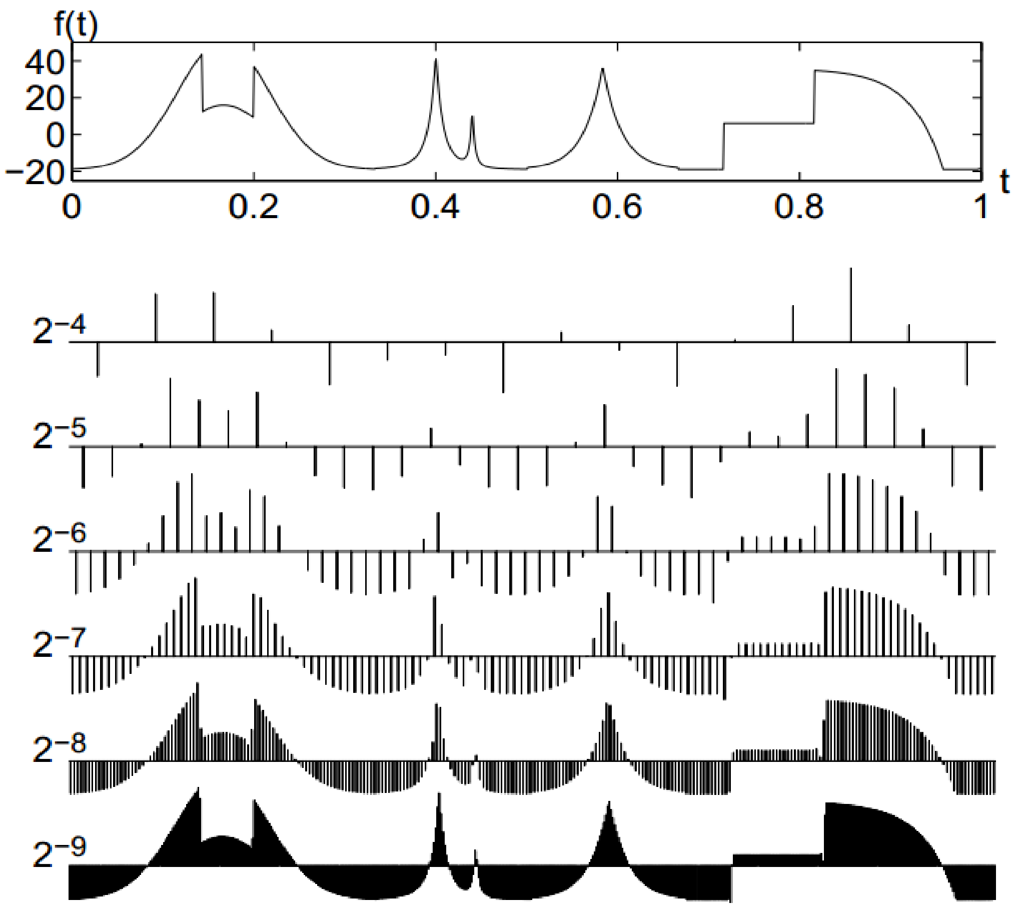
Approximation

Orthogonal projection of f over $\mathbf{V}_j$: $P_{\mathbf{V}_j}(f) = \sum_n \langle f, \phi_{j,n} \rangle \phi_{j,n}$

Approximation at the scale 2^j: $a_j[n] = \langle f, \phi_{j,n} \rangle$



Discrete multiresolution approximations $a_j[n]$ at scales 2^j, computed with cubic splines

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Scaling Equation

$\mathbf{V}_j \subset \mathbf{V}_{j-1} \Longrightarrow 2^{-1/2}\phi(t/2) \in \mathbf{V}_1 \subset \mathbf{V}_0$

$\{\phi(t-n)\}_{n\in\mathbb{Z}}$: orthonormal basis of $\mathbf{V}_0$

$$\frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n]\phi(t-n) \qquad h[n] = \left\langle \frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right), \phi(t-n) \right\rangle$$

Fourier Transform

$$\hat{\phi}(2\omega) = \frac{1}{\sqrt{2}}\hat{h}(\omega)\hat{\phi}(\omega) \qquad \hat{h}(\omega) = \sum_{n=-\infty}^{+\infty} h[n]e^{-in\omega}$$

$$\hat{\phi}(2^{-p+1}\omega) = \frac{1}{\sqrt{2}}\hat{h}(2^{-p}\omega)\hat{\phi}(2^{-p}\omega) \qquad p \geq 0$$

$$\hat{\phi}(\omega) = \left(\prod_{p=1}^{P} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}}\right)\hat{\phi}(2^{-P}\omega) \qquad \text{If } \hat{\phi}(\omega) \text{ is continuous at } \omega = 0$$

$$\Longrightarrow \lim_{P\to+\infty} \hat{\phi}(2^{-P}\omega) = \hat{\phi}(0) \qquad \hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}}\hat{\phi}(0)$$

Scaling Equation

Theorem 7.2: Mallat, Meyer. Let $\phi \in \mathbf{L}^2(\mathbb{R})$ be an integrable scaling function. The Fourier series of $h[n] = \langle 2^{-1/2}\phi(t/2), \phi(t-n)\rangle$ satisfies

$$\forall \omega \in \mathbb{R}, \qquad |\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2 \tag{7.29}$$

and

$$\hat{h}(0) = \sqrt{2} \tag{7.30}$$

Conversely, if $\hat{h}(\omega)$ is 2π periodic and continuously differentiable in a neighborhood of $\omega = 0$, if it satisfies (7.29) and (7.30) and if

$$\inf_{\omega \in [-\pi/2, \pi/2]} |\hat{h}(\omega)| > 0$$

then

$$\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}}$$

is the Fourier transform of a scaling function $\phi \in \mathbf{L}^2(\mathbb{R})$.

Wavelet bases

Scaling Equation

$$\forall \omega \in \mathbb{R}, \qquad \left|\hat{h}(\omega)\right|^2 + \left|\hat{h}(\omega+\pi)\right|^2 = 2 \qquad (7.29)$$

Discrete filters $h[n]$ satisfying (7.29) are called *conjugate mirror filters*.

It entirely determines the scaling function and most of its properties. The scaling function is compactly supported if and only if h has a finite number of non zero coefficients.

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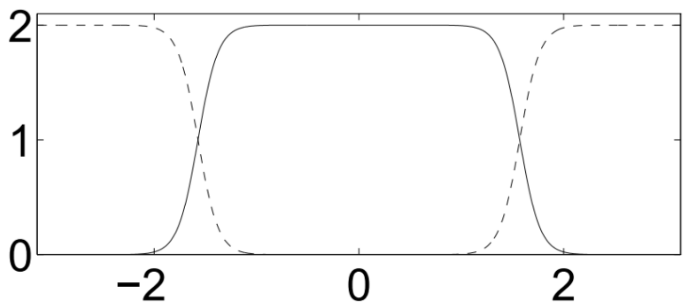
Example: Cubic Spline



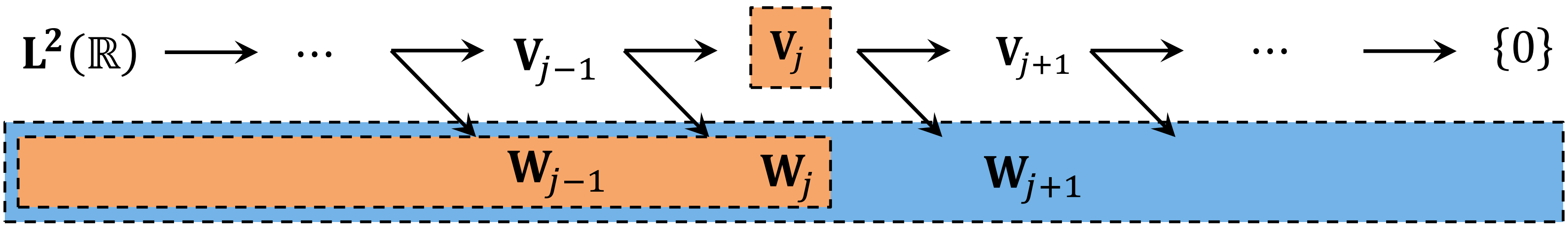
The solid line gives $|\hat{h}(\omega)|^2$ on $[-\pi, \pi]$ for a cubic spline multiresolution. The dotted line corresponds to $|\hat{g}(\omega)|^2$

n	h[n]
0	0,766130398
1,-1	0,433923147
2,-2	-0,050201753
3,-3	-0,110036987
4,-4	0,032080869
5,-5	0,042068328
6,-6	-0,017176331
7,-7	-0,017982291
8,-8	0,008685294
9,-9	0,008201477
10,-10	-0,004353840
11,-11	-0,003882426
12,-12	0,002186714
13,-13	0,001882120
14,-14	-0,001103748
15,-15	-0,000927187
16,-16	0,000559952
17,-17	0,000462093
18,-18	-0,000285414
19,-19	-0,000232304
20,-20	0,000146098

Conjugate Mirror Filters $h[n]$ for Cubic Splines $m = 3$. The coefficients below 10^{-4} are not given

Wavelet bases

Orthogonal Wavelets



Decomposition into detail spaces: $\mathbf{L}^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{+\infty} \mathbf{W}_j = \mathbf{V}_{j_0} \bigoplus_{j \le j_0} \mathbf{W}_j$

Wavelet basis of $\mathbf{L}^2(\mathbb{R})$: Full: $\{\psi_{j,n} | (j,n) \in \mathbb{Z}^2\}$

Truncated: $\{\psi_{j,n} | j \le j_0, n \in \mathbb{Z}\} \cup \{\phi_{j_0,n} | n \in \mathbb{Z}\}$

Wavelet basis of $\mathbf{L}^2(\mathbb{R}/\mathbb{Z})$: $\{\psi_{j,n} | j \le j_0, 0 \le n < 2^{-j}\} \cup \{\phi_{j_0,n} | 0 \le n < 2^{-j_0}\}$

Wavelet transform: computing the coefficients:

$$\{\langle f, \psi_{j,n} \rangle\}_{0 \le n < 2^{-j}}^{j \le j_0} \cup \{\langle f, \phi_{j_0,n} \rangle\}_{0 \le n < 2^{-j_0}}$$

Wavelet bases

Orthogonal Wavelets

Detail spaces $\mathbf{W}_j$: $\mathbf{V}_{j-1} = \mathbf{V}_j \oplus \mathbf{W}_j$ $\quad P_{\mathbf{V}_{j-1}}f = P_{\mathbf{V}_j}f + P_{\mathbf{W}_j}f$

$$
\begin{tikzcd}
\mathbf{L}^2(\mathbb{R}) \arrow[r] & \cdots \arrow[r] \arrow[dr] & \mathbf{V}_{j-1} \arrow[r] \arrow[dr] & \mathbf{V}_j \arrow[r] \arrow[dr] & \mathbf{V}_{j+1} \arrow[r] \arrow[dr] & \cdots \arrow[r] & \{0\} \\
& & \mathbf{W}_{j-1} & \mathbf{W}_j & \mathbf{W}_{j+1} & &
\end{tikzcd}
$$

Wavelet function ψ: $\quad \mathbf{W}_j = \mathrm{Span}\left(\psi_{j,n}\right)_{n\in\mathbb{Z}} \quad \psi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-2^j n}{2^j}\right)$

Theorem 7.3: Mallat, Meyer. Let ϕ be a scaling function and h the corresponding conjugate mirror filter. Let ψ be the function having a Fourier transform

$$\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$$

with

$$\hat{g}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi)$$

Let us denote

$$\psi_{j,n}(t) = \frac{1}{\sqrt{2^j}}\psi\left(\frac{t-2^j n}{2^j}\right)$$

For any scale 2^j, $\{\psi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{W}_j$. For all scales, $\{\psi_{j,n}\}_{(j,n)\in\mathbb{Z}^2}$ is an orthonormal basis of $\mathbf{L}^2(\mathbb{R})$.

Wavelet bases

Orthogonal Wavelets

Lemma 7.1. The family $\{\psi_{j,n}\}_{n\in\mathbb{Z}}$ is an orthonormal basis of $\mathbf{W}_j$ if and only if

$$|\hat{g}(\omega)|^2 + |\hat{g}(\omega+\pi)|^2 = 2$$

and

$$\hat{g}(\omega)\hat{h}^*(\omega) + \hat{g}(\omega+\pi)\hat{h}^*(\omega+\pi) = 0$$

$\hat{g}$: Fourier series of $g[n] = \left\langle \frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right), \phi(t-n)\right\rangle$

$$\Downarrow$$

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\,\phi(t-n)$$

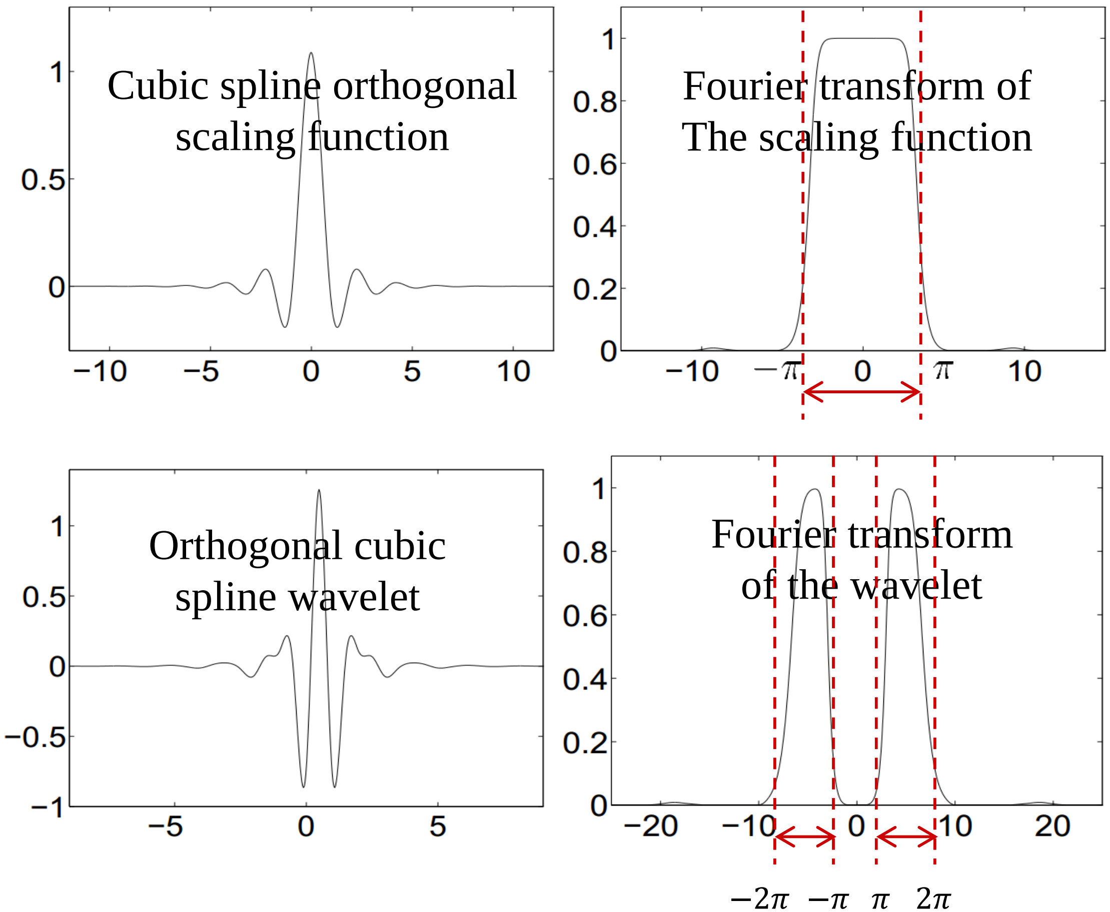
$$\hat{g}(\omega) = e^{-i\omega}\hat{h}^*(\omega+\pi) \xrightarrow[\textit{Fourier transform}]{\textit{Inverse}} g[n] = (-1)^{1-n}h[1-n]$$

The h and g filters are a conjugate mirror filter bank.

Wavelet bases

Orthogonal Wavelets

Cubic spline scaling function and the corresponding cubic spline Battle-Lemarié wavelet, and their Fourier transform. The wavelet is a cubic spline because it is a linear combination of cubic splines.

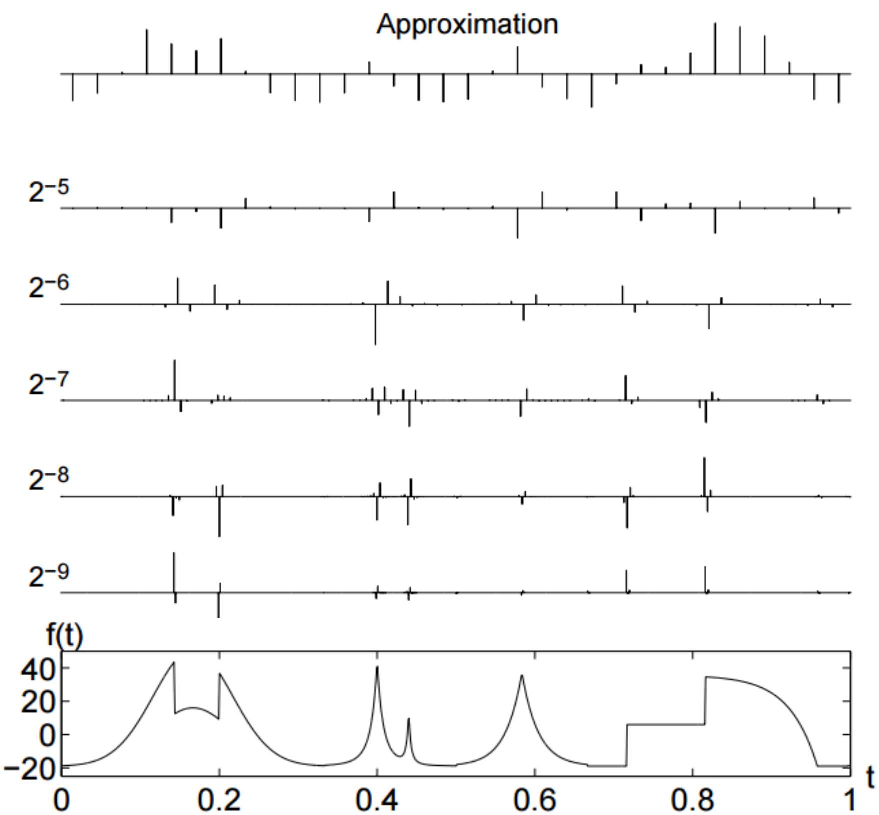


Wavelet bases

Orthogonal Wavelets

Orthogonal projection of a signal f in a detail space $\mathbf{W}_j$:

$$P_{\mathbf{W}_j} f = \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n} \quad \Longrightarrow \quad f = \sum_{j=-\infty}^{+\infty} P_{\mathbf{W}_j} f = \sum_{j=-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$



Wavelet coefficients $d_j[n] = \langle f, \psi_{j,n} \rangle$ calculated at scales 2^j with the cubic spline wavelet. The up or down Diracs give the amplitudes of positive or negative wavelet coefficients at a distance $2^j n$ at each scale 2^j. At the top is the remaining coarse-signal approximation $a_J[n] = \langle f, \phi_{J,n} \rangle$ for $J = -5$.

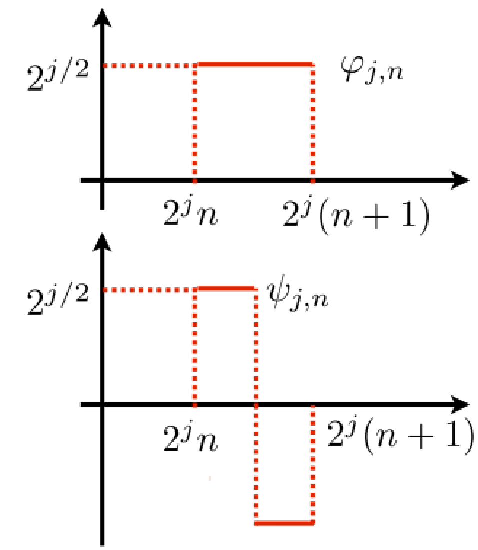
Wavelet bases

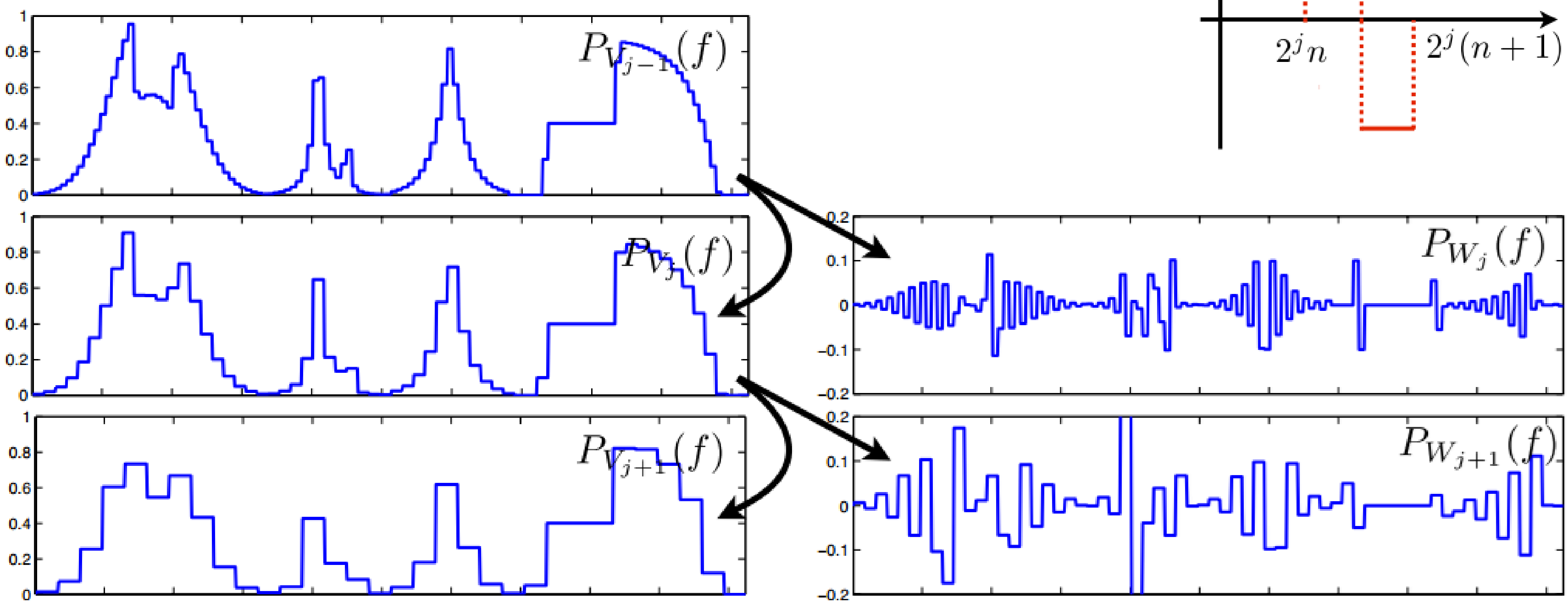
Haar Wavelets

Projection on approximate space $P_{V_j}(f) = \sum_n \langle f, \phi_{j,n} \rangle \phi_{j,n}$

Projection on detail space $P_{W_j}(f) = \sum_n \langle f, \psi_{j,n} \rangle \psi_{j,n}$

$$\underbrace{P_{V_{j-1}}(f)}_{\text{Fine approximation}} = \underbrace{P_{V_j}(f)}_{\text{Coarse approximation}} + \underbrace{P_{W_j}(f)}_{\text{Details}}$$





Wavelet Bases

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Wavelet bases

Choosing a Wavelet

- Most applications of wavelet bases exploit their ability to efficiently approximate particular classes of functions with few nonzero wavelet coefficients

- The design of ψ must be optimized to produce a maximum number of wavelet coefficients $\langle f, \psi_{j,n} \rangle$ that are close to zero. This depends on the regularity of f, the number of vanishing moments of ψ, and the size of its support

- To construct an appropriate wavelet from a conjugate mirror filter $h[n]$, we relate these properties to conditions on $\hat{h}(\omega)$

Wavelet bases

Property: Vanishing Moments

ψ has p vanishing moments if

$$\int_{-\infty}^{+\infty} t^k \psi(t)dt = 0 \quad \text{for} \quad 0 \leq k < p$$

This means that ψ is orthogonal to any polynomial of degree $p-1$.

If f is regular and ψ has enough vanishing moments, then the wavelet coefficients $|\langle f, \psi_{j,n} \rangle|$ are small at fine scales 2^j.

Wavelet bases

Property: Vanishing Moments

Theorem 7.4 relates the number of vanishing moments of ψ to the vanishing derivatives of $\hat{\psi}(\omega)$ at $\omega = 0$ and to the number of zeros of $\hat{h}(\omega)$ at $\omega = \pi$. It also proves that polynomials of degree $p-1$ are then reproduced by the scaling functions.

Theorem 7.4: *Vanishing Moments.* Let ψ and ϕ be a wavelet and a scaling function that generate an orthogonal basis. Suppose that $|\psi(t)| = O\left((1+t^2)^{-p/2-1}\right)$ and $|\phi(t)| = O\left((1+t^2)^{-p/2-1}\right)$. The four following statements are equivalent:

1. The wavelet ψ has p vanishing moments.
2. $\hat{\psi}(\omega)$ and its first $p-1$ derivatives are zero at $\omega = 0$.
3. $\hat{h}(\omega)$ and its first $p-1$ derivatives are zero at $\omega = \pi$.
4. For any $0 \leq k < p$,

$$q_k(t) = \sum_{n=-\infty}^{+\infty} n^k \phi(t-n) \text{ is a polynomial of degree } k$$

Wavelet bases

Property: Size of Support

If f has an isolated singularity at t_0 and if t_0 is inside the support of $\psi_{j,n}(t) = 2^{-j/2}\psi(2^{-j}t - n)$, then $\langle f, \psi_{j,n} \rangle$ may have a large amplitude. If ψ has a compact support of size K, at each scale 2^j there are K wavelets $\psi_{j,n}$ with a support including t_0

To minimize the number of high-amplitude coefficients we must reduce the support size of ψ. Theorem 7.5 relates the support size of h to the support of ϕ and ψ:

Theorem 7.5: *Compact Support.* The scaling function ϕ has a compact support if and only if h has a compact support and their supports are equal. If the support of h and ϕ is $[N_1, N_2]$, then the support of $\psi$is $[(N_1 - N_2 + 1)/2, (N_2 - N_1 + 1)/2]$.

Wavelet bases

Property: Support versus Moments

- Daubechies has proved that, to generate an orthogonal wavelet with p vanishing moment, a filter h with minimum length $2p$ had to be used

- Daubechies filters, which generate Daubechies wavelets, have a length of $2p$. Daubechies wavelets are optimal in the sense that they have a minimum size of support for a given number of vanishing moments

- When choosing a particular wavelet, we face a trade off between the number of vanishing moments and the support size:

f: few isolated singularities
very regular between singularities

↓

large number of vanishing moment

f: dense singularities

↓

decrease the size of support at the cost of reducing the number of vanishing moments

Wavelet bases

Property: Regularity

- Wavelet regularity is much less important than their vanishing moments. The number of vanishing moments and the regularity of orthogonal wavelets are related, but it is the number of vanishing moments and not the regularity that affects the amplitude of the wavelet coefficients at fine scales.

- Theorem 7.6 relates the uniform Lipschitz regularity of ϕ and ψ to the number of zeros of $\hat{h}(\omega)$ at $\omega = \pi$:

Theorem 7.6: *Tchamitchian.* Let $\hat{h}(\omega)$ be a conjugate mirror filter with p zeroes at π and that satisfies the sufficient conditions of Theorem 7.2. Let us perform the factorization

$$\hat{h}(\omega) = \sqrt{2}\left(\frac{1+e^{i\omega}}{2}\right)^p \hat{l}(\omega)$$

If $\sup_{\omega\in\mathbb{R}}\left|\hat{l}(\omega)\right| = B$, then ψ and ϕ are uniformly Lipschitz α for

$$\alpha < \alpha_0 = p - \log_2 B - 1$$

- There is no compactly supported orthogonal wavelet which is indefinitely differentiable

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Property: Symmetry

- Symmetric scaling functions and wavelets are important because they are used to build bases of regular wavelets over an interval, rather than the real axis.

- Daubechies has proved that, for a wavelet to be symmetric or antisymmetric, its filter must have a linear complex phase.

- The only symmetric compactly supported conjugate mirror filter is the Haar filter, which corresponds to a discontinuous wavelet with one vanishing moment. Besides the Haar wavelet, there is no symmetric compactly supported orthogonal wavelet.

Wavelet bases

Magnitude of Wavelet Coefficients

ψ has *p* vanishing moments if:

$$\int_{-\infty}^{+\infty} t^k \psi(t)dt = 0 \quad 0 \leq k < p$$

If f is C^α on $\mathrm{supp}(\psi_{j,n})$, $p \geq \alpha$: $t = \dfrac{x - 2^j n}{2^j}$

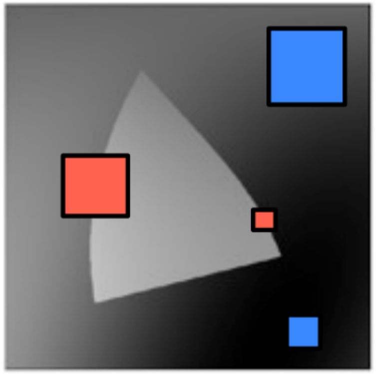
$$\langle f, \psi_{j,n} \rangle = \frac{1}{2^{j\frac{d}{2}}} \int f(x) \psi\left(\frac{x - 2^j n}{2^j}\right) dx = 2^{j\frac{d}{2}} \int R(2^j t) \psi(t) dt$$

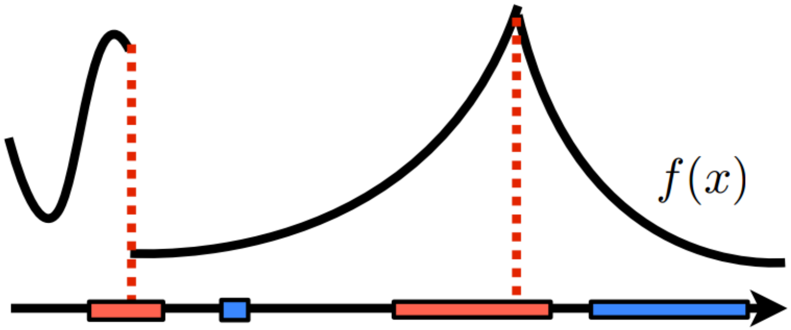
$$f(x) = P(x - 2^j n) + R(x - 2^j n) = P(2^j t) + R(2^j t)$$

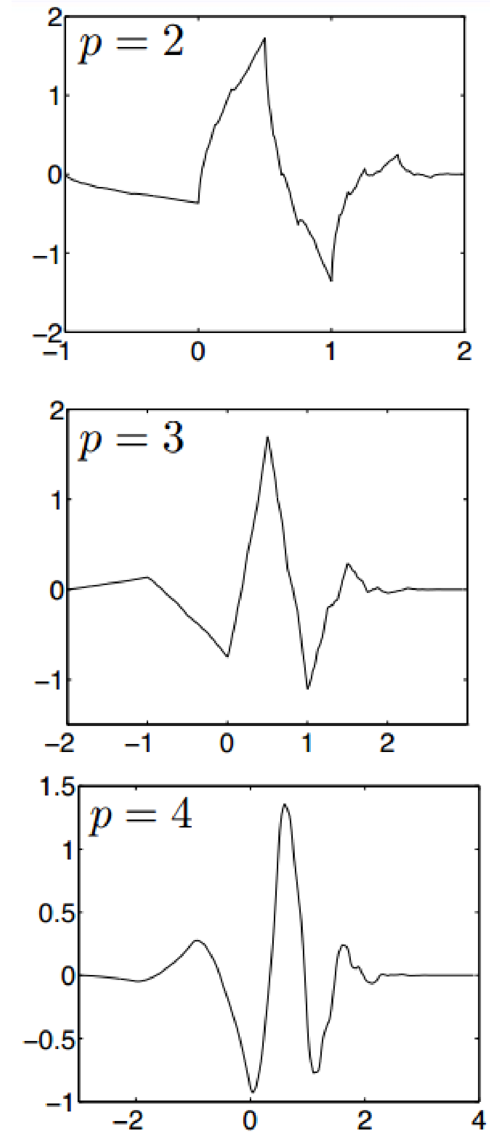
where $\deg(P) < \alpha$ and $|R(x)| \leq C_f ||x||^\alpha$

$$|\langle f, \psi_{j,n} \rangle| \leqslant C_f \|\psi\|_1 2^{j(\alpha + d/2)}$$

$$|\langle f, \psi_{j,n} \rangle| \leqslant \|f\|_\infty \|\psi\|_1 2^{j\frac{d}{2}}$$







Wavelet bases

Vanishing Moment Constraint

Smooth areas: vanishing moments:

$$\forall k \leq p-1, \quad \int_0^1 \psi(x) x^k dx = 0$$

$$\Longrightarrow \langle f, \psi_{j,n} \rangle \approx 0 \text{ if } f \text{ is } \mathrm{C}^\alpha, \alpha < p$$

$$\text{on } \mathrm{Supp}(\psi_{j,n})$$

Near singularities: $Supp(\psi_{j,n})$ small

$\Longrightarrow$ few large coefficients near singularities

Theorem: $\mathrm{Supp}(\psi_{j,n})$ is larger than $2p-1$

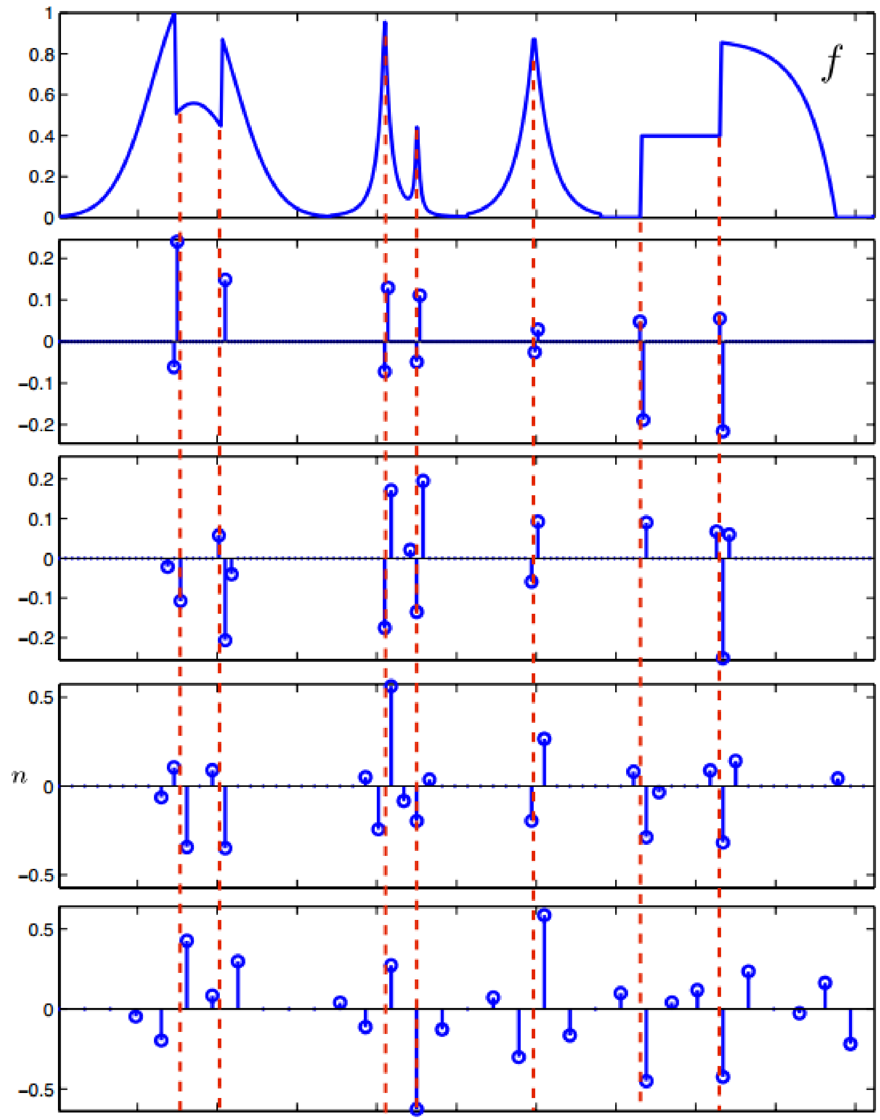
$$Smoothness\ of\ \psi: \ f_M = \sum_{(j,n) \in I_M} \langle f, \psi_{j,n} \rangle \psi_{j,n}$$

f_M same smoothness as ψ

$\Longrightarrow$ only for cosmetic reason

$Heuristic:$ increasing p increases

the smoothness of ψ



Wavelet bases

Daubechies Family

Compute h that satisfies

$$|\hat{h}(\omega)|^2 + |\hat{h}(\omega+\pi)|^2 = 2$$

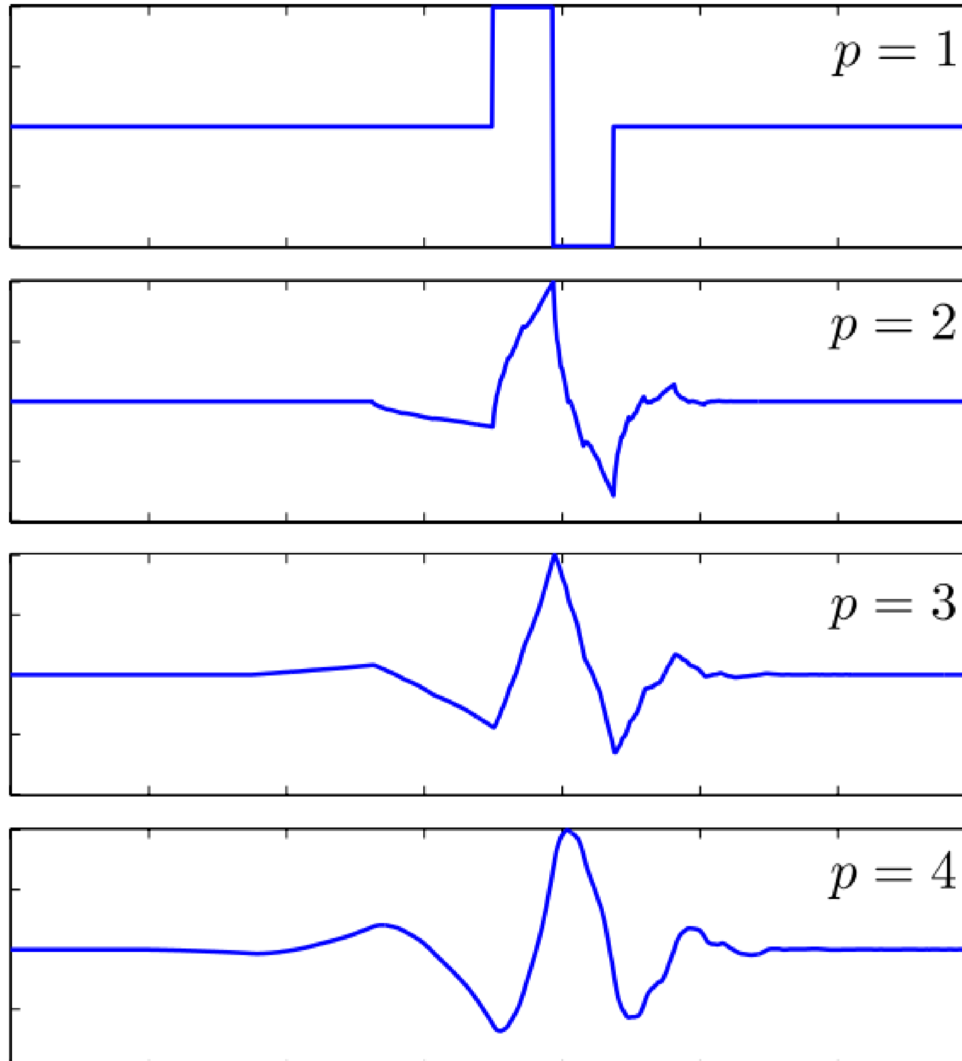
$$\hat{h}(0) = \sqrt{2}$$

$$p \iff \forall k < p, \frac{d^k \hat{h}}{d\omega^k}(\pi) = 0$$

Daubechies wavelet with p VM:

Orthogonal wavelets

Minimal support $2p-1$



Filter $h[n]$: $n=0$

$p=1$ (Haar): $[0.7071; 0.7071]$

$p=2$: $[0.4830; 0.8365; 0.2241; -0.1294]$

$n=0$ $n=0$

$p=3$: $[0; 0.3327; 0.8069; 0.4599; -0.1350; -0.0854; 0.0352]$

Wavelet Bases

- Orthogonal Wavelet Bases
- Classes of Wavelet Bases
- Wavelets and Filter Banks
- Biorthogonal Wavelet Bases

Wavelet bases

Orthogonal Wavelets and Discrete Filters

Conjugate mirror filters (*g* and *h*) are a particular instance of perfect reconstruction filter banks. The dyadic nature of multiresolution approximations are closely related to the possibility of implementing elementary signal subsampling by erasing one sample every two, and elementary oversampling by zero insertion between two consecutive samples.

The coefficients $a_1[n]$ in V_j and $d_1[n]$ in W_j are computed from $a_0[n]$ in V_{j-1} by applying conjugate mirror filters and subsampling the output:

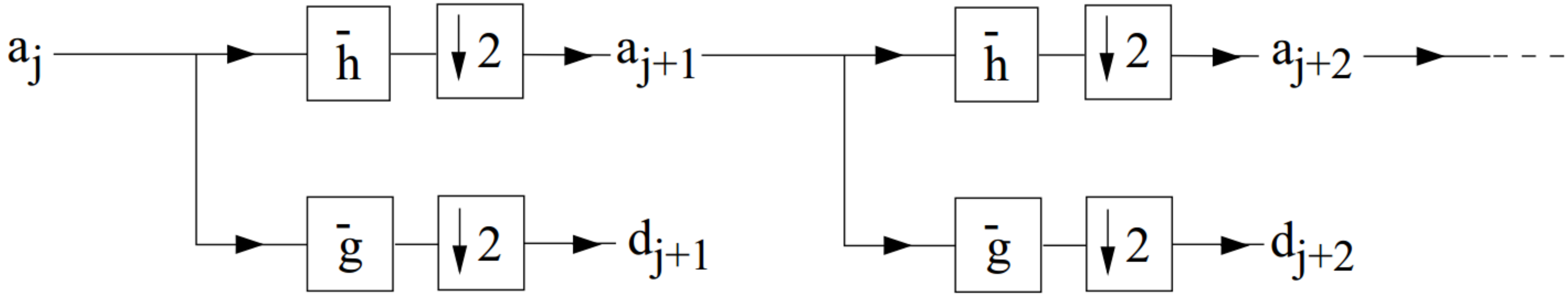
$$a_1[n] = a_0 * \bar{h}[2n] \qquad d_1[n] = a_0 * \bar{g}[2n]$$

$$\text{with} \quad \bar{h}[n] = h[-n] \qquad \bar{g}[n] = g[-n]$$

Wavelets and scaling functions are evaluated as in the orthogonal case.

Wavelet bases

Fast Wavelet Transform



The coefficients of h are defined by the scaling equation

$$\frac{1}{\sqrt{2}}\phi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} h[n]\phi(t-n)$$

or, in the Fourier domain $\hat{\phi}(\omega) = \prod_{p=1}^{+\infty} \frac{\hat{h}(2^{-p}\omega)}{\sqrt{2}} \hat{\phi}(0)$

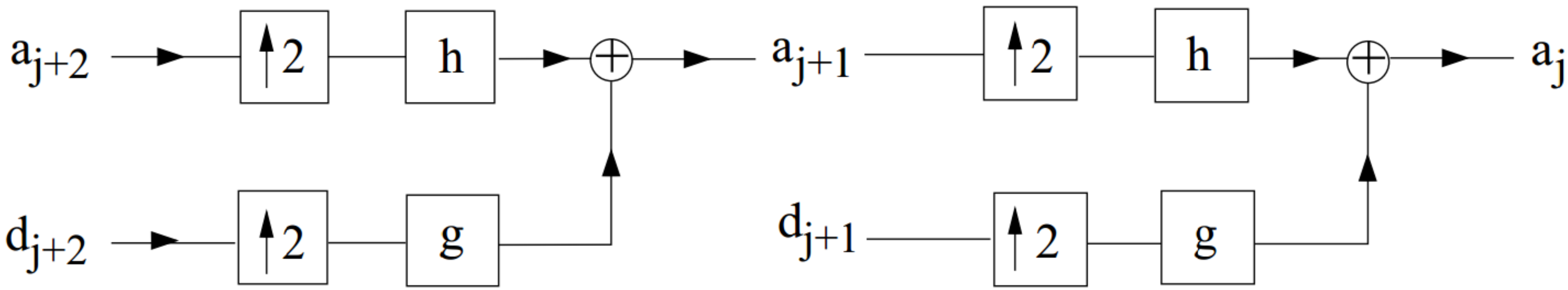
The coefficients of g are defined by the wavelet scaling equation

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\phi(t-n)$$

or, in the Fourier domain $\hat{\psi}(\omega) = \frac{1}{\sqrt{2}}\hat{g}\left(\frac{\omega}{2}\right)\hat{\phi}\left(\frac{\omega}{2}\right)$

Wavelet bases

Fast Inverse Wavelet Transform



$a_0[n]$ is reconstructed from $a_1[n]$ and $d_1[n]$ by inserting zeroes between two consecutive samples and summing their convolutions with *h* and *g*:

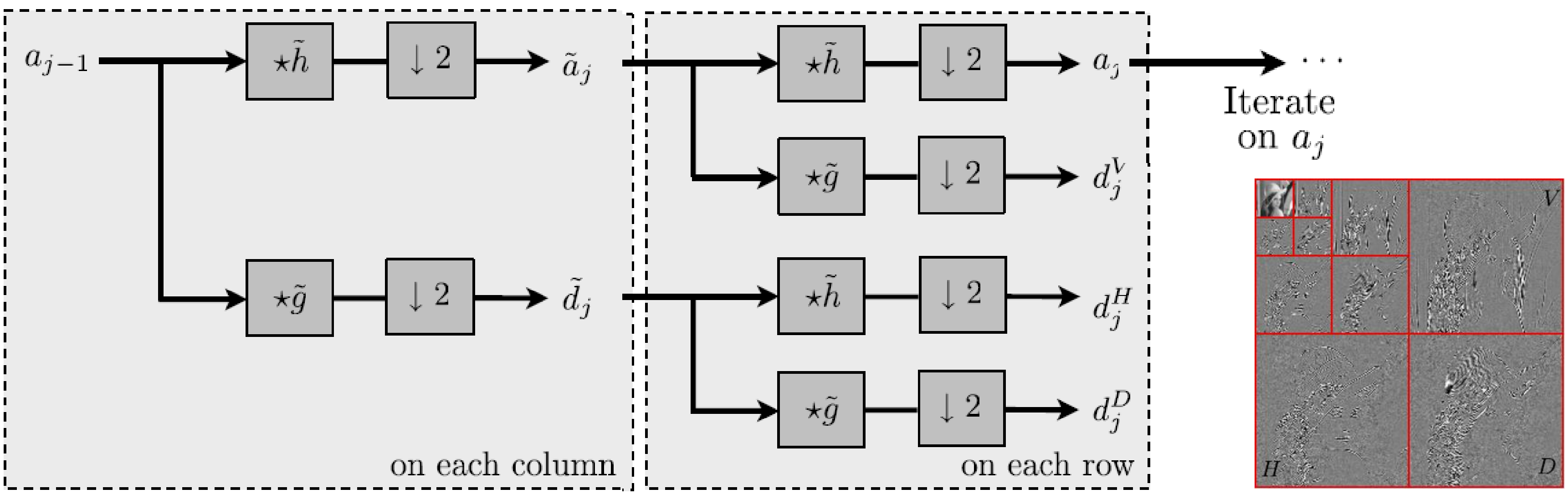
$$a_0[n] = z(a_1) * h[n] + z(d_1) * g[n]$$

where the *z* operator represents the insertion of zeroes.

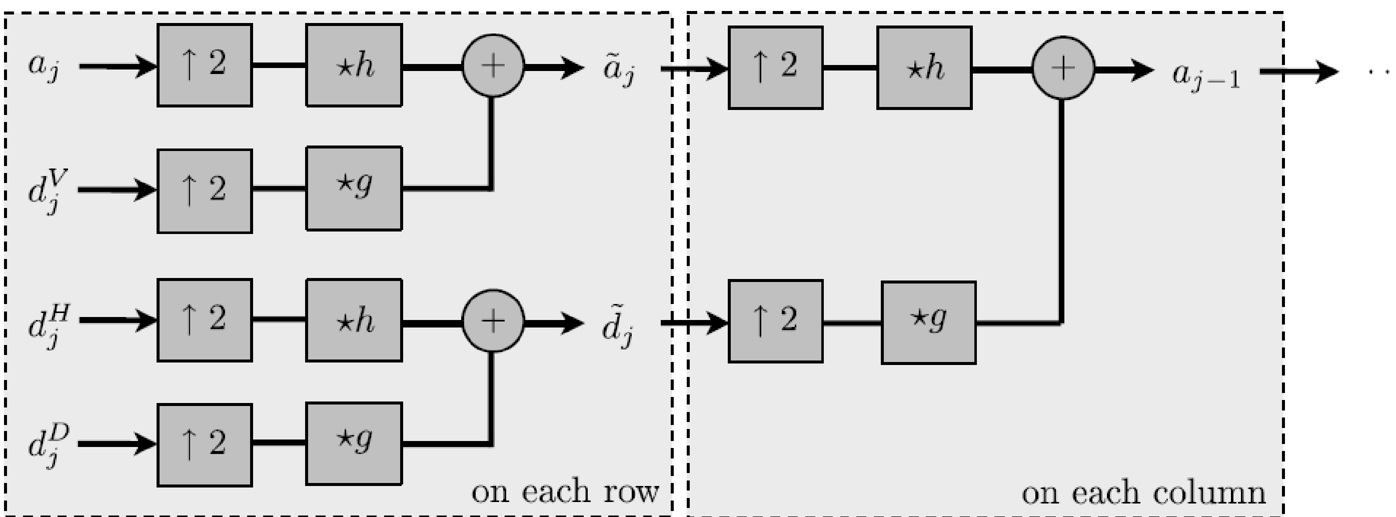
The construction of orthogonal wavelets is equivalent to the synthesis of conjugate mirror filters having a stability property.

Wavelet bases

Fast 2D Wavelet Transform



Fast Inverse 2D Wavelet Transform



Wavelet bases

Perfect Reconstruction Filter Banks

A perfect reconstruction filter bank decomposes a signal by filtering and subsampling. It reconstructs it by inserting zeroes, filtering and summation.

A (discrete) two-channel multi-rate filter bank convolves a signal a_0 with a low-pass filter $\bar{h}[n] = h[-n]$ and a high-pass filter $\bar{g}[n] = g[-n]$ and then subsamples by 2 the output:

$$a_1[n] = a_0 \star \bar{h}[2n] \quad \text{and} \quad d_1[n] = a_0 \star \bar{g}[2n]$$

A reconstructed signal $\tilde{a}_0$ is obtained by filtering the zero expanded signals with a dual low-pass filter $\tilde{h}$ and a dual high-pass filter $\tilde{g}$. If $\check{x}$ denotes the signal obtained from x by inserting a zero between every sample, this can be written as:

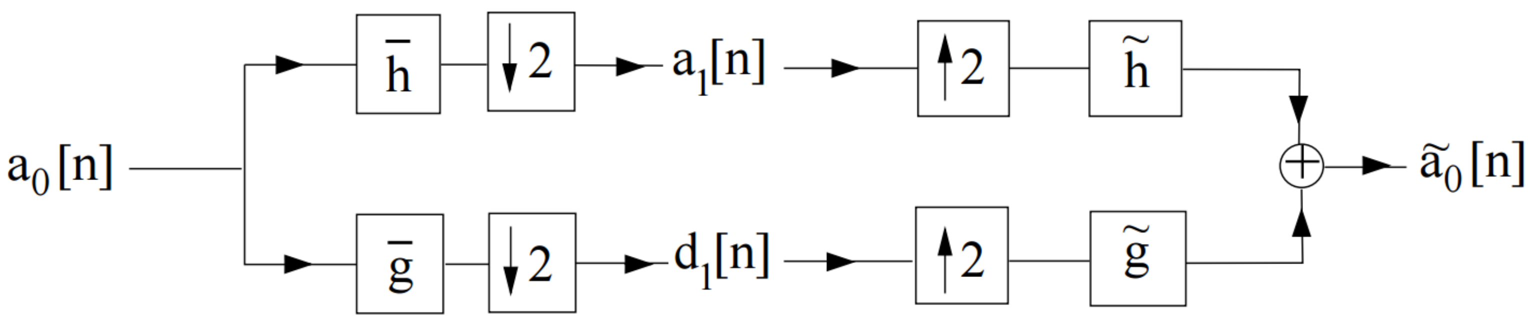
$$\tilde{a}_0[n] = \check{a}_1 \star \tilde{h}[n] + \check{d}_1 \star \tilde{g}[n].$$

$$\check{x}[n] = \begin{cases} x[p] & \text{if } n = 2p \\ 0 & \text{if } n = 2p+1 \end{cases}$$

Wavelet bases

Perfect Reconstruction Filter Banks

The following figure illustrates the decomposition and reconstruction process. The input signal is filtered by a low-pass and a high-pass filter and subsampled. The reconstruction is performed by inserting zeroes and filtering with dual filters $\tilde{h}$ and $\tilde{g}$



The filter bank is said to be a *perfect reconstruction filter bank* when $\tilde{a}_0 = a_0$.

If, additionally, $h = \tilde{h}$ and $g = \tilde{g}$, the filters are called *conjugate mirror filters*.

Wavelet bases

Perfect Reconstruction Filter Banks

Perfect reconstruction filter banks are characterized in a theorem by Theorem 7.11:

Theorem 7.11: Vetterli. The filter bank performs an exact reconstruction for any input signal if and only if

$$\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega+\pi)\hat{\tilde{g}}(\omega)=0$$

and

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{g}^*(\omega)\hat{\tilde{g}}(\omega)=2$$

Eliminating g and $\tilde{g}$ leads to the necessary condition

$$\hat{h}^*(\omega)\hat{\tilde{h}}(\omega)+\hat{h}^*(\omega+\pi)\hat{\tilde{h}}(\omega+\pi)=2$$

For finite impulse-response filters, there exists $a \in \mathbb{R}$ and $l \in \mathbb{Z}$ such that

$$\hat{g}(\omega)=a e^{-i(2l+1)\omega}\hat{\tilde{h}}^*(\omega+\pi) \quad \text{and} \quad \hat{\tilde{g}}(\omega)=a^{-1}e^{-i(2l+1)\omega}\hat{h}^*(\omega+\pi)$$

Wavelet bases

Conjugate Mirror Filters

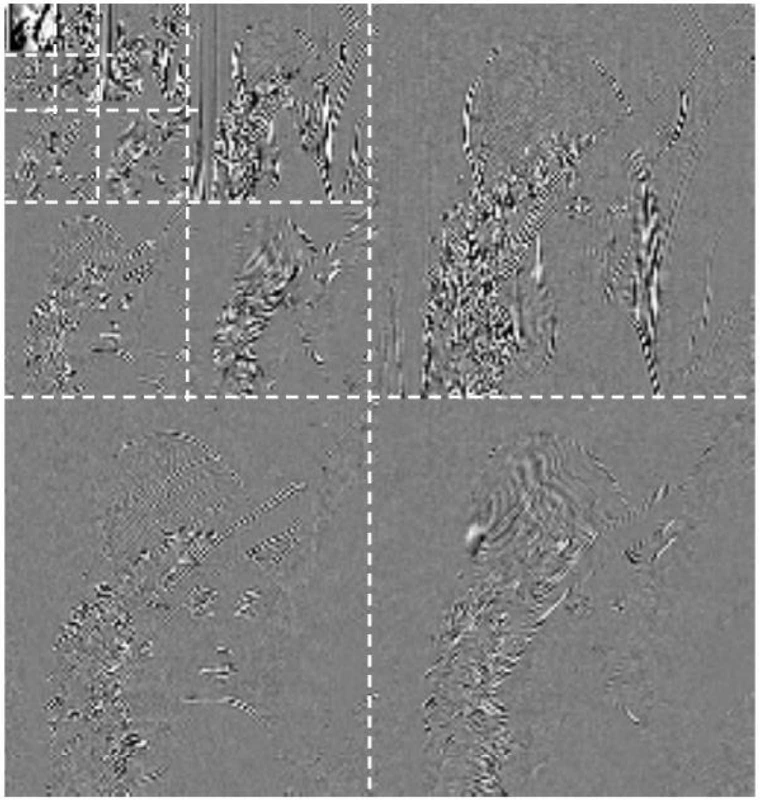
A finite impulse conjugate mirror filter bank is characterized by a filter h which satisfies:

$$\left|\hat{h}(\omega)\right|^2+\left|\hat{h}(\omega+\pi)\right|^2=2 \tag{7.139}$$

It is identical to the filter condition (7.29) in Theorem 7.2 that is required in order to synthesize orthogonal wavelets. It is also equivalent to discrete orthogonality properties:

A Riesz basis is orthonormal if the dual basis is the same as the original basis. For filter banks this means that $h=\tilde{h}$ and $g=\tilde{g}$. The filter h is then a conjugate mirror filter satisfying (7.139). And the resulting family $\{h[n-2l], g[n-2l]\}_{l\in\mathbb{Z}}$ is an orthogonal basis of $l^2(\mathbb{Z})$.

Example: pyramid decomposition



Original
$128, 129, 125, 64, 65, \cdots$

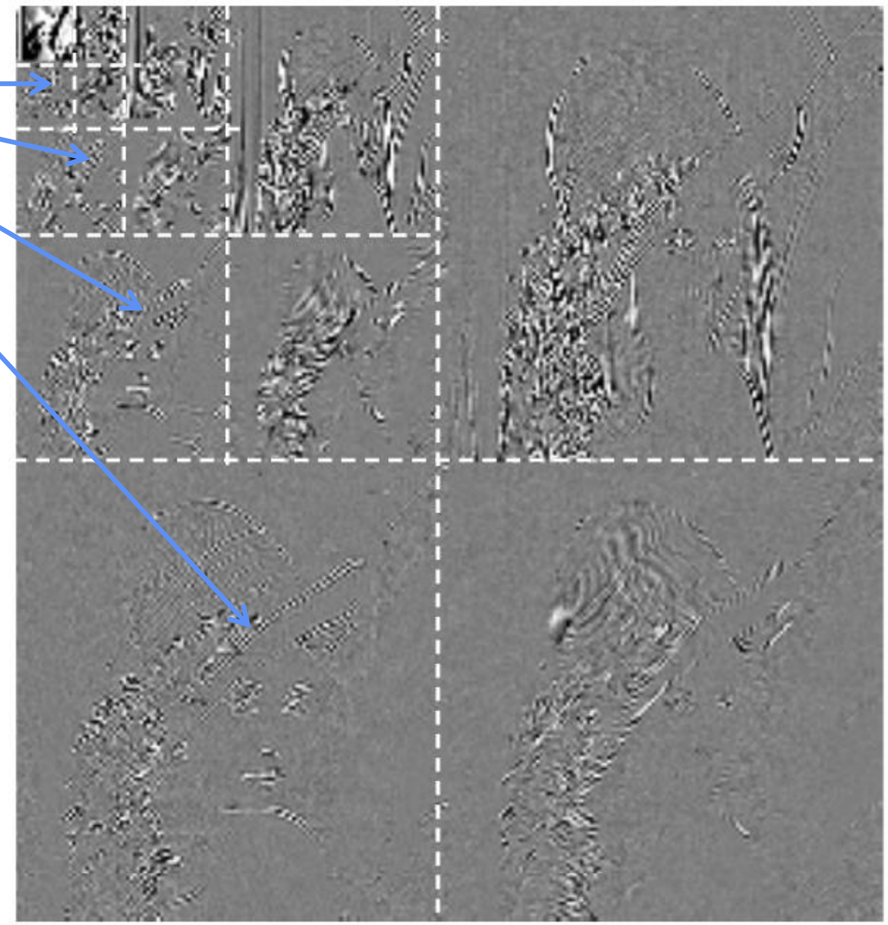
Transform coefficients
$4123, -12.4, -96.7, 4.5, \ldots$

Time meets frequency

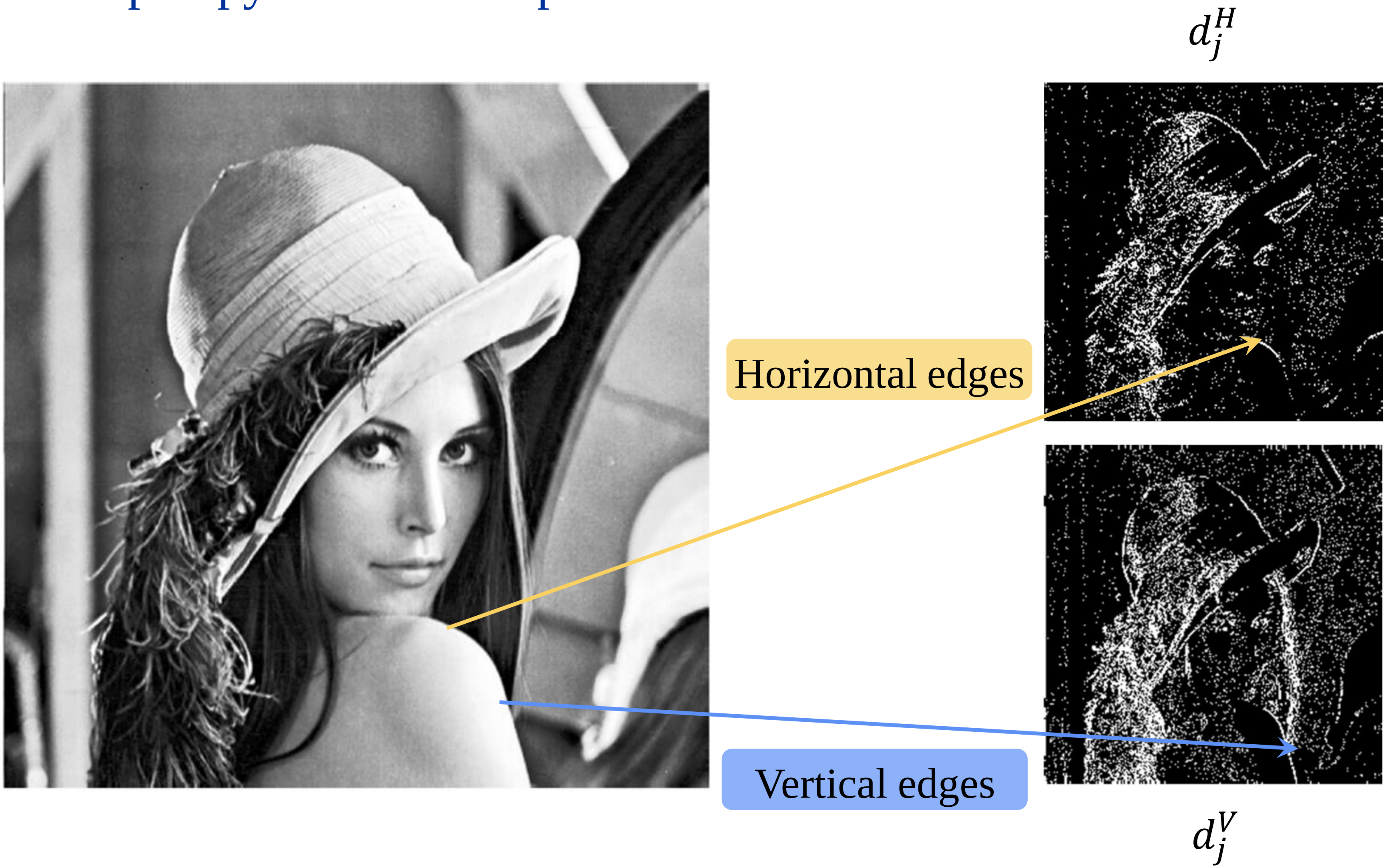
Example: pyramid decomposition

Similar data distribution on different scales

Exponentially degrades on the same spot



Example: pyramid decomposition



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Wavelet Bases

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Wavelet bases

Orthogonality and Biorthogonality

- When the Riesz basis is an orthogonal basis, the multiresolution approximation is orthogonal, and the base atom is called a scaling function. It is always possible to orthogonalize a multiresolution approximation.

- However, orthogonalities imposes some constraints that may not be desirable. One of the most important is that a compactly supported (orthogonal) scaling function cannot symmetric and continuous. The symmetry is useful in the analysis of finite signals.

- Some of these restrictions (notably the absence of symmetry) can be avoided by using biorthogonal multiresolution approximations.

Wavelet bases

Biorthogonal Multiresolution Approximations

A pair $\{\mathbf{V}_j\}_{j\in\mathbb{Z}}$, $\{\mathbf{V}_l^*\}_{l\in\mathbb{Z}}$ of multiresolution approximations is a biorthogonal multiresolution system if and only if

$$\mathbf{L}^2(\mathbb{R}) = \mathbf{V}_0 \oplus (\mathbf{V}_0^*)^{\perp}$$

Then $\mathbf{V}_0^*$ has a Riesz basis of the form $\theta^*(t-n), n \in \mathbb{Z}$, such that the translations of θ and of θ^* form a biorthogonal system:

$$\langle \theta(t-k), \theta^*(t-n) \rangle = \delta_{n-k}$$

We have a biorthogonal bases system instead of a single orthogonal basis.

Wavelet bases

Biorthogonal wavelets

Biorthogonal wavelets are defined similarly to orthogonal wavelets, except that the starting point is biorthogonal multiresolution approximations. The following decompositions are performed:

$$V_{j-1} = V_j \bigoplus W_j \text{ with } W_j \subset (V_j^+)^\perp$$

$$V_{j-1}^+ = V_j^+ \bigoplus W_j^+ \text{ with } W_j^+ \subset (V_j)^\perp$$

Like in the orthogonal case, a signal in $\mathbf{L}^2$ can be written as:

$$f(t) = \sum_{j,n\in\mathbb{Z}} \langle f, \psi_{j,n}^* \rangle \psi_{j,n}(t) \ = \sum_{n\in\mathbb{Z}} \langle f, \phi_{j,n}^* \rangle \phi_{j,n}(t) + \sum_{k\leq j,n\in\mathbb{Z}} \langle f, \psi_{k,n}^* \rangle \psi_{k,n}(t)$$

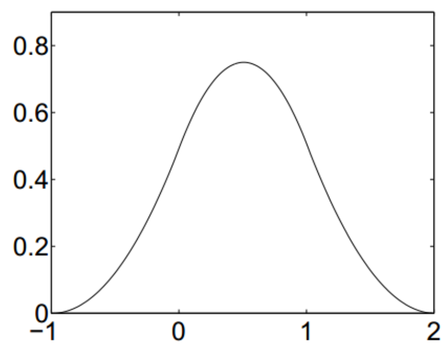
$$= \sum_{j,n\in\mathbb{Z}} \langle f, \psi_{j,n} \rangle \psi_{j,n}^*(t) \ = \sum_{n\in\mathbb{Z}} \langle f, \phi_{j,n} \rangle \phi_{j,n}^*(t) + \sum_{k\leq j,n\in\mathbb{Z}} \langle f, \psi_{k,n} \rangle \psi_{k,n}^*(t)$$

Wavelet bases

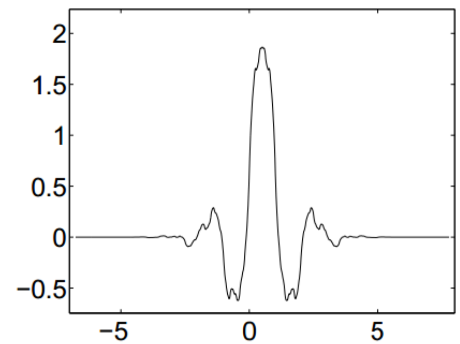
Biorthogonal wavelets

Below is a biorthogonal system which includes a cubic B-spline. Dropping the orthogonality constraint makes possible to have both regularity and symmetry.

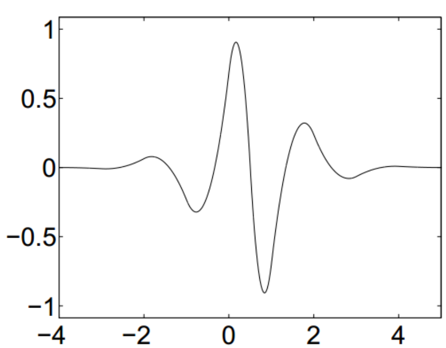
Biorthogonal cubic B-spline scaling function



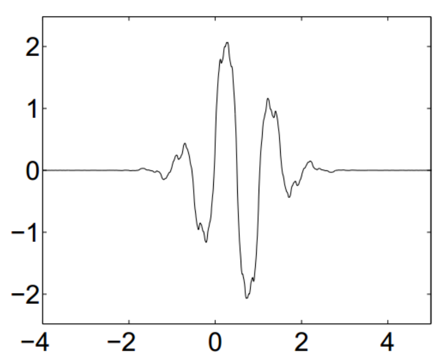
Dual scaling function



Biorthogonal spline wavelet



Dual Wavelet



Wavelet bases

Biorthogonal wavelets and Discrete Filters

The scaling equations on the scaling functions and wavelets show that the decomposition and reconstruction of a signal from a resolution to the next one is implemented by perfect reconstruction filter banks.

The coefficients $a_1[n]$ in V_j and $d_1[n]$ in W_j are computed from $a_0[n]$ in V_{j-1} by applying conjugate mirror filters and subsampling the output:

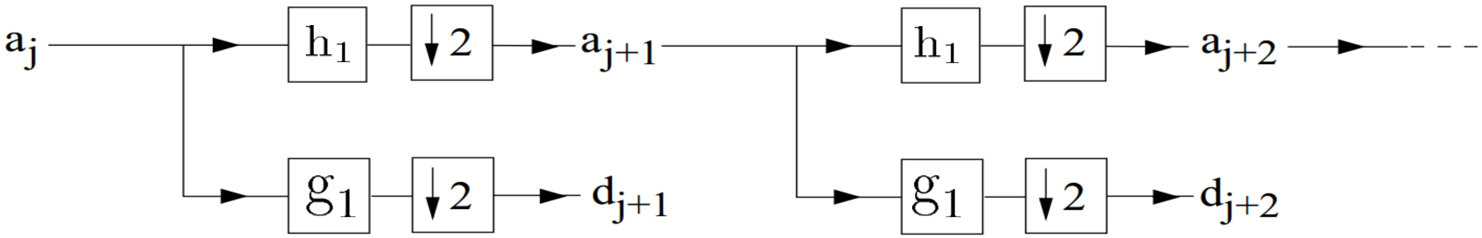
$$a_1[n] = a_0 * h_1[2n] \qquad d_1[n] = a_0 * g_1[2n]$$

$$\text{with} \quad h_1[n] = h[-n] \qquad g_1[n] = g[-n]$$

The construction of biorthogonal wavelets is equivalent to the synthesis of perfect reconstruction filters having a stability property.

Wavelet bases

Fast wavelet transform:



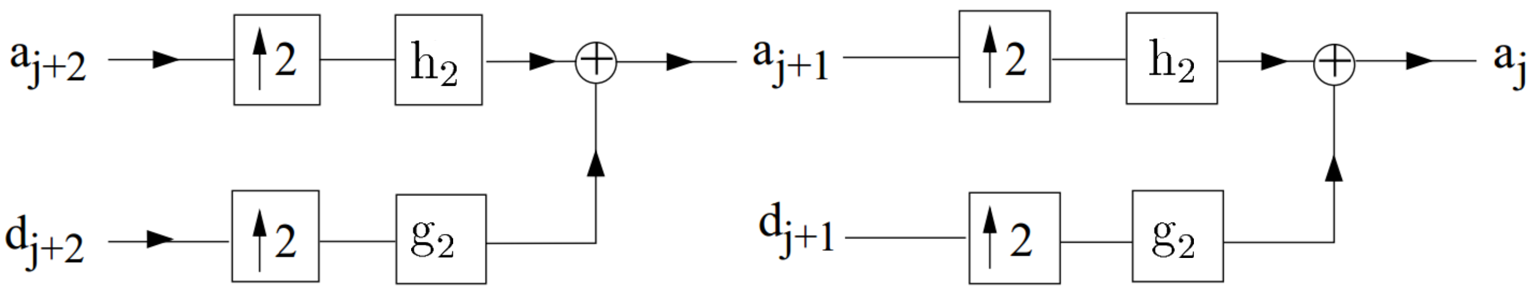
The coefficients of *h* and *g* are defined by the scaling equations

$$\phi^{+}(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} h[n]\phi^{+}(2t-n)$$

$$\psi^{+}(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} g[n]\phi^{+}(2t-n)$$

Wavelet bases

Fast inverse wavelet transform:



$a_0[n]$ is reconstructed from $a_1[n]$ and $d_1[n]$ by inserting zeroes between two consecutive samples and summing their convolutions with the dual filters h_2 and g_2 which define the dual scaling equations:

$$a_0[n] = z(a_1) * h_2[n] + z(d_1) * g_2[n]$$

where the z operator represents the insertion of zeroes.

The coefficients of h_2 and g_2 are defined by the scaling equations:

$$\phi(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} h_2[n]\phi(2t-n) \qquad \psi(t) = \sqrt{2} \sum_{n=-\infty}^{+\infty} g_2[n]\phi(2t-n)$$

Wavelet bases

Property: Scaling Equation

As in the orthogonal case, $\psi(t)$ and $\phi(t/2)$ are related by a scaling equation which is a consequence of the inclusions of the resolution spaces from coarse to fine:

$$\frac{1}{\sqrt{2}}\psi\left(\frac{t}{2}\right) = \sum_{n=-\infty}^{+\infty} g[n]\phi(t-n)$$

Similar equations exist for the dual functions which determine the filters h_2 and g_2.

Wavelet bases

Property: Vanishing Moments

A biorthogonal wavelet has m vanishing moments if and only if its dual scaling function generates polynomials up to degree m. This can be verified by looking at the biorthogonal decomposition formulas.

Hence there is an equivalence theorem between vanishing moments and the number of zeroes of the filter's transfer, provided that duality has to be taken into account. Thus the following three properties are equivalent:

- the wavelet ψ has p vanishing moments
- the dual scaling function ϕ_2 generates polynomials up to degree p
- the transfer function of the dual filter h_2 and it $p-1$ first derivatives vanish at $\omega = \pi$

and the dual result is also valid. Duality appears naturally, because the filters determine the degree of the polynomials which can be generated by the scaling function, and this degree is equal to the number of vanishing moments of the dual wavelet.

Wavelet bases

Property: Compact Support

If the filters h and h_2 have a finite support, then the scaling functions have the same support, and the wavelets are compactly supported. If the supports of the scaling functions are respectively $[N_1, N_2]$ and $[M_1, M_2]$, then the corresponding wavelets have support $[(N_1 - M_2 + 1)/2, (N_2 - M_1 + 1)/2]$ and $[(M_1 - N_2 + 1)/2, (M_2 - N_1 + 1)]$ respectively.

The atoms are thus compactly supported if and only if the filters h and h_2 are.

Wavelet bases

Property: Regularity

Theorem 7.6 provides again a sufficient regularity condition. Remember that this condition bears on the filter h which determines the scaling equation. Hence the regularity of the primal atoms are related to the primal filters.

Theorem 7.6: *Tchamitchian.* Let $\hat{h}(\omega)$ be a conjugate mirror filter with p zeroes at π and that satisfies the sufficient conditions of Theorem 7.2. Let us perform the factorization

$$\hat{h}(\omega) = \sqrt{2}\left(\frac{1+e^{i\omega}}{2}\right)^p \hat{l}(\omega)$$

If $\sup_{\omega\in\mathbb{R}}\left|\hat{l}(\omega)\right| = B$, then ψ and ϕ are uniformly Lipschitz α for

$$\alpha < \alpha_0 = p - \log_2 B - 1$$

Wavelet bases

Property: Wavelet Balancing

Consider the following decomposition of f:

$$f = \sum_{n,j=-\infty}^{+\infty} \langle f, \psi_{j,n}^{*} \rangle \psi_{j,n}$$

The number of vanishing moments of a wavelet is determined by its dual filter. It corresponds to the approximating power of the dual multiresolution sequence. This is why it is preferred to synthesize a decomposition filter h with many vanishing moments, and possibly with a small support.

On the other hand, this same filter h determines the regularity of ϕ, and hence of ψ. This regularity increases with the number of vanishing moments, that is, with the number of zeroes of h.

Wavelet bases

Property: Symmetry

Unlike the orthogonal case, it is possible to synthesize biorthogonal wavelets and scaling functions which are symmetric or antisymmetric and compactly supported. This makes it possible to use the folding technique to build wavelets on an interval.

If the filters h and h_2 have and odd length and are symmetric with respect to 0, then the scaling functions have an even length and are symmetric, and the wavelets are also symmetric. If the filters have an even length and are symmetric with respect to $n = 1/2$, then the scaling functions are symmetric with respect to $n = 1/2$, while the wavelets are antisymmetric.

Homework

Chapter 7: 7.2 and 7.7 (a) (b) (A Wavelet Tour of Signal Processing, 3rd edition)

Q & A

Many Thanks